seen with the *Mars Climate Orbiter*. This probe was launched by NASA on December 11, 1998. On September 23, 1999, while attempting to guide the probe into its planned orbit around Mars, NASA lost contact with it. Subsequent investigations showed a piece of software called SM_FORCES (or "small forces") was recording thruster performance data in the English units of pound-seconds (lb-s). However, other pieces of software that used these values for course corrections expected them to be recorded in the SI units of newton-seconds (N-s), as dictated in the software interface protocols. This error caused the probe to follow a very different trajectory from what NASA thought it was following, which most likely caused the probe either to burn up in the Martian atmosphere or to shoot out into space. This failure to pay attention to unit conversions cost hundreds of millions of dollars, not to mention all the time invested by the scientists and engineers who worked on the project.

**1.4 Check Your Understanding** Given that 1 lb (pound) is 4.45 N, were the numbers being output by SM_FORCES too big or too small?

# 1.4 | Dimensional Analysis

## Learning Objectives

By the end of this section, you will be able to:

- Find the dimensions of a mathematical expression involving physical quantities.
- Determine whether an equation involving physical quantities is dimensionally consistent.

The **dimension** of any physical quantity expresses its dependence on the base quantities as a product of symbols (or powers of symbols) representing the base quantities. Table 1.3 lists the base quantities and the symbols used for their dimension. For example, a measurement of length is said to have dimension L or $\mathrm{L}^1$, a measurement of mass has dimension M or $\mathrm{M}^1$, and a measurement of time has dimension T or $\mathrm{T}^1$. Like units, dimensions obey the rules of algebra. Thus, area is the product of two lengths and so has dimension $\mathrm{L}^2$, or length squared. Similarly, volume is the product of three lengths and has dimension $\mathrm{L}^3$, or length cubed. Speed has dimension length over time, L/T or $\mathrm{LT}^{-1}$. Volumetric mass density has dimension $\mathrm{M/L}^3$ or $\mathrm{ML}^{-3}$, or mass over length cubed. In general, the dimension of any physical quantity can be written as $\mathrm{L}^a\mathrm{M}^b\mathrm{T}^c\mathrm{I}^d\Theta^e\mathrm{N}^f\mathrm{J}^g$ for some powers $a, b, c, d, e, f,$ and $g$. We can write the dimensions of a length in this form with $a = 1$ and the remaining six powers all set equal to zero: $\mathrm{L}^1 = \mathrm{L}^1\mathrm{M}^0\mathrm{T}^0\mathrm{I}^0\Theta^0\mathrm{N}^0\mathrm{J}^0$. Any quantity with a dimension that can be written so that all seven powers are zero (that is, its dimension is $\mathrm{L}^0\mathrm{M}^0\mathrm{T}^0\mathrm{I}^0\Theta^0\mathrm{N}^0\mathrm{J}^0$) is called **dimensionless** (or sometimes "of dimension 1," because anything raised to the zero power is one). Physicists often call dimensionless quantities *pure numbers*.

| Base Quantity | Symbol for Dimension |
|---|---|
| Length | L |
| Mass | M |
| Time | T |
| Current | I |
| Thermodynamic temperature | Θ |
| Amount of substance | N |
| Luminous intensity | J |

**Table 1.3 Base Quantities and Their Dimensions**

Physicists often use square brackets around the symbol for a physical quantity to represent the dimensions of that quantity. For example, if $r$ is the radius of a cylinder and $h$ is its height, then we write $[r] = \mathrm{L}$ and $[h] = \mathrm{L}$ to indicate the dimensions of the radius and height are both those of length, or L. Similarly, if we use the symbol $A$ for the surface area of a cylinder and $V$ for its volume, then $[A] = \mathrm{L}^2$ and $[V] = \mathrm{L}^3$. If we use the symbol $m$ for the mass of the cylinder and $\rho$

for the density of the material from which the cylinder is made, then $[m] = \text{M}$ and $[\rho] = \text{ML}^{-3}$.

The importance of the concept of dimension arises from the fact that any mathematical equation relating physical quantities must be **dimensionally consistent**, which means the equation must obey the following rules:

- Every term in an expression must have the same dimensions; it does not make sense to add or subtract quantities of differing dimension (think of the old saying: "You can't add apples and oranges"). In particular, the expressions on each side of the equality in an equation must have the same dimensions.
- The arguments of any of the standard mathematical functions such as trigonometric functions (such as sine and cosine), logarithms, or exponential functions that appear in the equation must be dimensionless. These functions require pure numbers as inputs and give pure numbers as outputs.

If either of these rules is violated, an equation is not dimensionally consistent and cannot possibly be a correct statement of physical law. This simple fact can be used to check for typos or algebra mistakes, to help remember the various laws of physics, and even to suggest the form that new laws of physics might take. This last use of dimensions is beyond the scope of this text, but is something you will undoubtedly learn later in your academic career.

## Example 1.4

### Using Dimensions to Remember an Equation

Suppose we need the formula for the area of a circle for some computation. Like many people who learned geometry too long ago to recall with any certainty, two expressions may pop into our mind when we think of circles: $\pi r^2$ and $2\pi r$. One expression is the circumference of a circle of radius $r$ and the other is its area. But which is which?

### Strategy

One natural strategy is to look it up, but this could take time to find information from a reputable source. Besides, even if we think the source is reputable, we shouldn't trust everything we read. It is nice to have a way to double-check just by thinking about it. Also, we might be in a situation in which we cannot look things up (such as during a test). Thus, the strategy is to find the dimensions of both expressions by making use of the fact that dimensions follow the rules of algebra. If either expression does not have the same dimensions as area, then it cannot possibly be the correct equation for the area of a circle.

### Solution

We know the dimension of area is $\text{L}^2$. Now, the dimension of the expression $\pi r^2$ is

$$[\pi r^2] = [\pi] \cdot [r]^2 = 1 \cdot \text{L}^2 = \text{L}^2,$$

since the constant $\pi$ is a pure number and the radius $r$ is a length. Therefore, $\pi r^2$ has the dimension of area. Similarly, the dimension of the expression $2\pi r$ is

$$[2\pi r] = [2] \cdot [\pi] \cdot [r] = 1 \cdot 1 \cdot \text{L} = \text{L},$$

since the constants $2$ and $\pi$ are both dimensionless and the radius $r$ is a length. We see that $2\pi r$ has the dimension of length, which means it cannot possibly be an area.

We rule out $2\pi r$ because it is not dimensionally consistent with being an area. We see that $\pi r^2$ is dimensionally consistent with being an area, so if we have to choose between these two expressions, $\pi r^2$ is the one to choose.

### Significance

This may seem like kind of a silly example, but the ideas are very general. As long as we know the dimensions of the individual physical quantities that appear in an equation, we can check to see whether the equation is dimensionally consistent. On the other hand, knowing that true equations are dimensionally consistent, we can match expressions from our imperfect memories to the quantities for which they might be expressions. Doing this will not help us remember dimensionless factors that appear in the equations (for example, if you had accidentally conflated the two expressions from the example into $2\pi r^2$, then dimensional analysis is no help), but it does help us remember the correct basic form of equations.

**1.5 Check Your Understanding** Suppose we want the formula for the volume of a sphere. The two expressions commonly mentioned in elementary discussions of spheres are $4\pi r^2$ and $4\pi r^3/3$. One is the volume of a sphere of radius *r* and the other is its surface area. Which one is the volume?

## Example 1.5

### Checking Equations for Dimensional Consistency

Consider the physical quantities $s, \; v, \; a,$ and $t$ with dimensions $[s] = \text{L}, \; [v] = \text{LT}^{-1}, \; [a] = \text{LT}^{-2}$, and $[t] = \text{T}$. Determine whether each of the following equations is dimensionally consistent: (a) $s = vt + 0.5at^2$; (b) $s = vt^2 + 0.5at$; and (c) $v = \sin(at^2/s)$.

### Strategy

By the definition of dimensional consistency, we need to check that each term in a given equation has the same dimensions as the other terms in that equation and that the arguments of any standard mathematical functions are dimensionless.

### Solution

a. There are no trigonometric, logarithmic, or exponential functions to worry about in this equation, so we need only look at the dimensions of each term appearing in the equation. There are three terms, one in the left expression and two in the expression on the right, so we look at each in turn:

$$[s] = \text{L}$$
$$[vt] = [v] \cdot [t] = \text{LT}^{-1} \cdot \text{T} = \text{LT}^0 = \text{L}$$
$$[0.5at^2] = [a] \cdot [t]^2 = \text{LT}^{-2} \cdot \text{T}^2 = \text{LT}^0 = \text{L}.$$

All three terms have the same dimension, so this equation is dimensionally consistent.

b. Again, there are no trigonometric, exponential, or logarithmic functions, so we only need to look at the dimensions of each of the three terms appearing in the equation:

$$[s] = \text{L}$$
$$[vt^2] = [v] \cdot [t]^2 = \text{LT}^{-1} \cdot \text{T}^2 = \text{LT}$$
$$[at] = [a] \cdot [t] = \text{LT}^{-2} \cdot \text{T} = \text{LT}^{-1}.$$

None of the three terms has the same dimension as any other, so this is about as far from being dimensionally consistent as you can get. The technical term for an equation like this is *nonsense*.

c. This equation has a trigonometric function in it, so first we should check that the argument of the sine function is dimensionless:

$$\left[\frac{at^2}{s}\right] = \frac{[a] \cdot [t]^2}{[s]} = \frac{\text{LT}^{-2} \cdot \text{T}^2}{\text{L}} = \frac{\text{L}}{\text{L}} = 1.$$

The argument is dimensionless. So far, so good. Now we need to check the dimensions of each of the two terms (that is, the left expression and the right expression) in the equation:

$$[v] = \text{LT}^{-1}$$
$$\left[\sin\left(\frac{at^2}{s}\right)\right] = 1.$$

The two terms have different dimensions—meaning, the equation is not dimensionally consistent. This equation is another example of “nonsense.”

### Significance

If we are trusting people, these types of dimensional checks might seem unnecessary. But, rest assured, any textbook on a quantitative subject such as physics (including this one) almost certainly contains some equations with typos. Checking equations routinely by dimensional analysis save us the embarrassment of using an incorrect equation. Also, checking the dimensions of an equation we obtain through algebraic manipulation is a great way to make sure we did not make a mistake (or to spot a mistake, if we made one).

**1.6 Check Your Understanding** Is the equation $v = at$ dimensionally consistent?

One further point that needs to be mentioned is the effect of the operations of calculus on dimensions. We have seen that dimensions obey the rules of algebra, just like units, but what happens when we take the derivative of one physical quantity with respect to another or integrate a physical quantity over another? The derivative of a function is just the slope of the line tangent to its graph and slopes are ratios, so for physical quantities $v$ and $t$, we have that the dimension of the derivative of $v$ with respect to $t$ is just the ratio of the dimension of $v$ over that of $t$:

$$\left[\frac{dv}{dt}\right] = \frac{[v]}{[t]}.$$

Similarly, since integrals are just sums of products, the dimension of the integral of $v$ with respect to $t$ is simply the dimension of $v$ times the dimension of $t$:

$$\left[\int v dt\right] = [v] \cdot [t].$$

By the same reasoning, analogous rules hold for the units of physical quantities derived from other quantities by integration or differentiation.

# 1.5 | Estimates and Fermi Calculations

## Learning Objectives

By the end of this section, you will be able to:

- Estimate the values of physical quantities.

On many occasions, physicists, other scientists, and engineers need to make *estimates* for a particular quantity. Other terms sometimes used are *guesstimates*, *order-of-magnitude approximations*, *back-of-the-envelope calculations*, or *Fermi calculations*. (The physicist Enrico Fermi mentioned earlier was famous for his ability to estimate various kinds of data with surprising precision.) Will that piece of equipment fit in the back of the car or do we need to rent a truck? How long will this download take? About how large a current will there be in this circuit when it is turned on? How many houses could a proposed power plant actually power if it is built? Note that estimating does not mean guessing a number or a formula at random. Rather, **estimation** means using prior experience and sound physical reasoning to arrive at a rough idea of a quantity's value. Because the process of determining a reliable approximation usually involves the identification of correct physical principles and a good guess about the relevant variables, estimating is very useful in developing physical intuition. Estimates also allow us perform "sanity checks" on calculations or policy proposals by helping us rule out certain scenarios or unrealistic numbers. They allow us to challenge others (as well as ourselves) in our efforts to learn truths about the world.

Many estimates are based on formulas in which the input quantities are known only to a limited precision. As you develop physics problem-solving skills (which are applicable to a wide variety of fields), you also will develop skills at estimating. You develop these skills by thinking more quantitatively and by being willing to take risks. As with any skill, experience helps. Familiarity with dimensions (see **Table 1.3**) and units (see **Table 1.1** and **Table 1.2**), and the scales of base quantities (see **Figure 1.4**) also helps.

To make some progress in estimating, you need to have some definite ideas about how variables may be related. The following strategies may help you in practicing the art of estimation:

- *Get big lengths from smaller lengths.* When estimating lengths, remember that anything can be a ruler. Thus, imagine breaking a big thing into smaller things, estimate the length of one of the smaller things, and multiply to get the length of the big thing. For example, to estimate the height of a building, first count how many floors it has. Then, estimate how big a single floor is by imagining how many people would have to stand on each other's